**3D Problems with Spherically Symmetric Potentials**

So we want to solve the equation:



where V(r) is presumed a general, but spherically symmetric, potential. Projecting onto the position + spin basis (remember our wavefunction exists in spin space too now ☺)



Now let’s take advantage of some symmetry. The potential is spherically symmetric, and so the radial degree of freedom of the Hamiltonian ought to ‘de-couple’ from the angular degrees of freedom – meaning the Hamiltonian ought to decouple into a radial part and an angular part. So we want to change independent variables to spherical coordinates.



Making the change of variables, the Laplacian goes to:



So filling this into our Hamiltonian we have:



Now we recognize the position representation of the angular momentum operator (squared), and define the radial momentum operator as:



and with these identifications we may profitably write:



H written in this form separates the kinetic energy of the particle into radial kinetic energy (the first term) and angular kinetic energy (the second term). To aid in this recognition you may recall that KEA = L2/2I = L2/2mr2 since in this case I = mr2. OK, now since the r-coordinate is independent of the angular coordinates, is independent of the spin quantum number we may use separation of variables to write out eigenfunction as:



where **χ** = (χ1,χ2) for short. It should be no surprise that Yℓm(θ,φ) is part of our solution, because a spherically Hamiltonian commutes with L2, i.e.,



and so H and L2 have simultaneous eigenfunctions. Filling this in we get:



So now we have to solve just a radial ODE to get the energy levels. Let’s fill in what *p*r is…pr2 = [r -1∂r r][r -1∂r r] = r-1∂r2 r.



and in this form it may occur to us to make a change of variables:



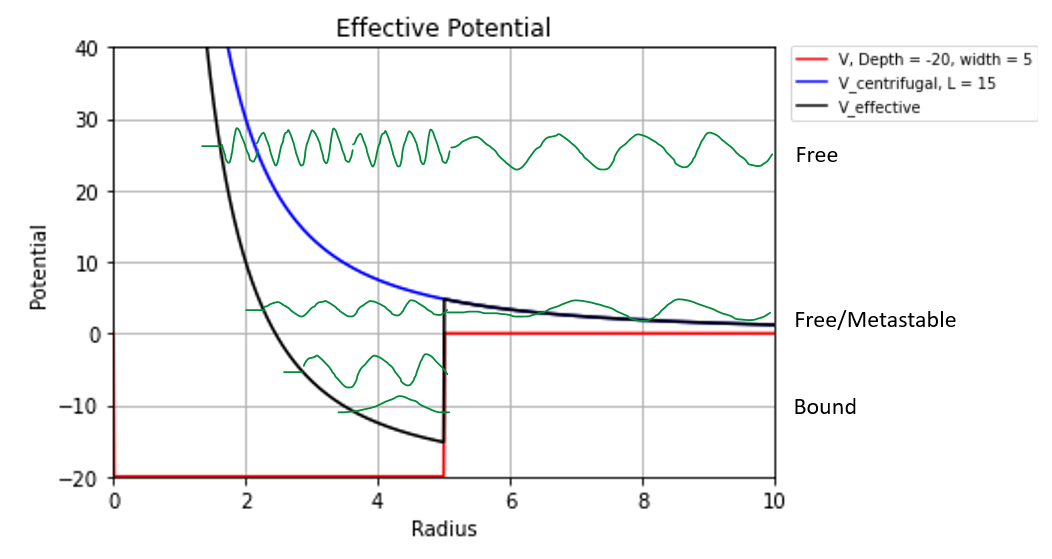
Doing so we have:



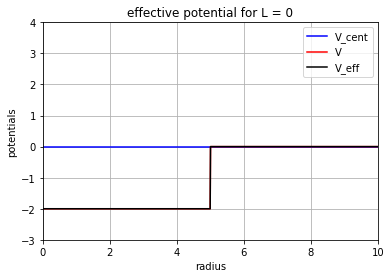
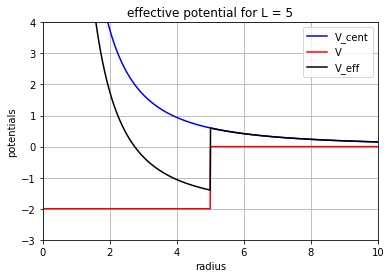
So this is an effective 1D (radial) Schrodinger equation, with an effective potential:

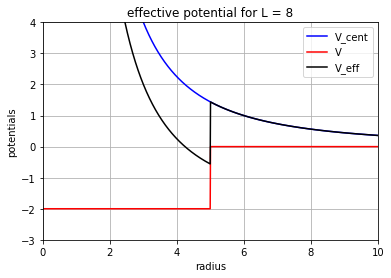
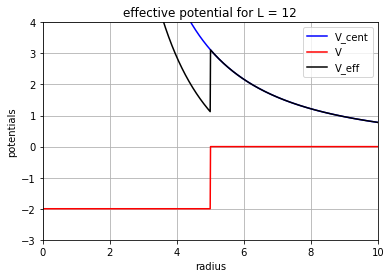


The first term is called the ‘centrifugal’ potential, though it is not a ‘real’ potential, as the centrifugal force is not a real force. Consider a step potential of depth V0 = -20 and radius R = 5, and a centrifugal potential with ℏ = 1, and m = 1 for simplicity, and ℓ = 15. I made a picture of some of the states we might expect to be accomodated at the various energy ranges.



We see that we can possibly have bound states ε < 0, metastable states ε > 0 which somewhat resonate between the walls of the centrifugal barrier, and totally free states. Note that not any state with energy less than the centrifugal barrier at r = R will be a metastable state – it has to be such that a whole number of wavelengths fits within the barrier (at least that would be the semiclassical criterion). A positive energy particle initially localized between those Veff potential humps would have a relatively long lifetime before it eventually leaked out into the rightward region, to freedom. This is the principle behind the phenomenon of radioative decay. If we lower ℓ, then the centrifugal potential diminishes, Veff 🡪 V, and we have much more room for bound states. As we increase ℓ however, the effective potential dominates V, making it harder and harder to accommodate bound states, as the negative effective potential region shrinks. Eventually, Veff will be entirely positive for all r. In that case, we will have no more possibility for bound states; however, metastable states will still be possible thanks to the fact that the effective potential will still have that wedge shape for r < R. Nonetheless, as ℓ increases, this wedge will get narrower and narrower, meaning any particle stuck in there would have higher and higher kinetic energy, and so making it much less likely that it will be stuck for any appreciable length of time. I plotted some potential for different ℓ’s below.

Can see we’d probably get lots of bound states for ℓ = 0, but no metastable states. We’d get fewer bound states for ℓ = 5, but now we’d be able to have metastable states. For ℓ = 9, we have barely any accomodation for bound states, but more for metastable states. For ℓ = 12, bound states are no longer possible, and even metastable states are becoming untenable.